Literature Seminar in Geometry **Spring 2023**

Exotic 7-Spheres

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Outline

Using a construction related to the quaternionic Hopf fibration, we construct all S^3 bundles over S^4, which are classified by $\mathbb{Z}^2 = \pi_3(\mathrm{SO}(4))$, and labeled as $M_{h,j}$. Using Reeb's theorem from Morse theory, we show that the $M_{h,j}$ satisfying $h + j = 1$ are homeomorphic to topological 7-spheres. Assuming these 7-spheres are diffeomorphic to the standard 7-sphere, we construct an associated 8-manifold and calculate its first Pontryagin class. Applying the Hirzebruch signature theorem, we can obtain a contradiction to the assumption that the $M_{h,j}$ are diffeomorphic to the standard 7-sphere, and we have our first explicit examples of exotic spheres.

We largely follow Milnor's original paper,[1] with additional exposition and clarifying material taken from a UChicago REU Paper[2] by Rachel McEnroe and some MathOverflow comments from Greg Kuperberg.[3]

[1] Milnor, "On Manifolds Homeomorphic to the 7-Sphere".

[2] McEnroe, "Milnor's Construction of Exotic 7-Spheres".

[3] Kuperberg, *Characteristic classes of sphere bundles over spheres in terms of clutching functions*; Kuperberg, *Morse Theory and Exotic Spheres*.

Background

Reeb's Theorem

Here we'll just discuss some results that we won't have time to prove in full generality. The first one is Reeb's theorem on spheres.

Theorem 2.1: Reeb

For M^n closed, if there exists a differentiable function $f : M^n \to \mathbb{R}$ having only two non-degenerate critical points x_0, x_1, then M^n is homeomorphic to S^n via a map which is a diffeomorphism except possibly at a single point.

The assumption of non-degeneracy of the critical poincts can in fact be omitted but this is more than we'll need.

The basic idea is that since M^n is compact, x_0 is (WLOG) the global maximum of f and x_1 is the global minimum so the handlebody decomposition thereof is the union of a 0-cell and an n-cell, which is a (topological) sphere.

Spheres on Spheres

Lemma 2.2

The S^3-bundles on S^4 are classified by $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z}^2$.

The construction here is that every sphere S^n can be covered by two charts (the complements of the North and South poles respectively), and we can force any fiber bundle to trivialize on these charts (since they are diffeomorphic to $\mathbb{R}^n$), so the nontrivial data of the fiber bundle is contained in the transition map between these charts. This map's domain is the intersection of the charts which is homotopic to S^{n-1} (the equator), and valued in the group of self-diffeomorphisms of the fiber S^k, which when $k = 3$ is homotopy equivalent to $\mathrm{O}(4)$, and thus the homotopy groups of the group of self-diffeomorphisms of the fiber classify all such fiber bundles. We reduce from $\mathrm{O}(4)$ to $\mathrm{SO}(4)$ since $\mathrm{O}(n)$ is disconnected and the homotopy groups demand basepoints.

It is straightforward to recover the transition map explicitly:

Corollary 2.3

Given a diffeomorphism $g : S^{n-1} \to S^{n-1}$, a transition map from $\mathbb{R}^n \setminus \{0\}$ to itself is given by

$$u \mapsto v = \frac{1}{\|u\|} g\left(\frac{u}{\|u\|}\right)$$

The resulting manifold is homeomorphic to S^n, with Morse function

$$f(x) = \frac{\|u\|^2}{1 + \|u\|^2} = \frac{1}{1 + \|v\|^2}$$

The fact that $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z}^2$ comes from the fact that $S^3 \times S^3$ double covers $\mathrm{SO}(4)$, and from the homotopy lifting lemma and the mapping lifting theorems, as we will discuss a little more in the next section.

It turns out that this classification by clutching works in all dimensions for *vector bundles* with structure group $\mathrm{GL}_k \mathbb{R}$, i.e, the k-vector bundles over S^n are classified by $\pi_{n-1}(\mathrm{GL}_k \mathbb{R})$.

The failure of this flavor of classification for 4+ dimensional *spherical* fibers is related to the Smale conjecture: since $\mathrm{Diff}\, S^n$ deformation retracts to $\mathrm{O}(n+1)$ for $n = 1, 2, 3$, $\pi_*(\mathrm{Diff}\, S^n) \cong \pi_*(\mathrm{SO}(n+1))$ (since $\mathrm{O}(n)$ has two components, we choose a basepoint) and this group classifies the S^n-bundles over any sphere, and the reduction of structure group from $\mathrm{GL}_k \mathbb{R}$ to $\mathrm{SO}(k)$ (impressionistically, i.e, I haven't filled in all the details) corresponds to taking the unit sphere subbundle.

Thus, all such S^n-bundles over any sphere arise as the unit sphere subbundle of some vector bundle in these low dimensions. The failure of the higher dimensional Smale conjecture corresponds to both "exotic" self-diffeomorphisms of these higher dimensional spheres and corresponding "exotic" sphere bundles not arising from any vector bundle. A fun little rabbit hole that I encountered when I incorrectly claimed that $\pi_{n-1}(\mathrm{SO}(k+1))$ classified all S^k-bundles over S^n.

Either way, Milnor dodges this entire issue by just enforcing $\mathrm{SO}(4)$ as the structure group for the bundles from the outset.

Main Result

Quaternionic Hopf Fibrations

The inspiration for our construction is the ordinary Hopf fibration, which is an S^1 bundle over S^2 with total space S^3 most easily realized in terms of $\mathbb{C}^2$ and $\mathbb{CP}^1$:

$$\begin{array}{ccc} S^1 = \mathrm{U}(1) & \hookrightarrow & S^3 \subseteq \mathbb{C}^2 \\ & & \big\downarrow {\scriptstyle (z_1,z_2)\mapsto[z_1,z_2]} \\ & & S^2 = \mathbb{CP}^1 \end{array}$$

Note that by the above discussion, S^1 bundles over S^2 should be classified by $\pi_1(\mathrm{SO}(2)) = \pi_1(S^1) = \mathbb{Z}$ and in fact we may obtain all such bundles in a way analogous to the construction central to our main result: let $g_i : S^1 \to \mathrm{SO}(2)$ be given by $g_i(z) \cdot v = z^n v$ where we are regarding S^1 as the unit complex numbers, which is a more quotidian identification than S^3 with the unit quaternions.

Our construction is based on the natural quaternionic analogue:

$$\begin{array}{ccc} S^3 = \mathrm{Spin}(3) & \hookrightarrow & S^7 \subseteq \mathbb{H}^2 \\ & & \big\downarrow {\scriptstyle (z_1,z_2)\mapsto[z_1,z_2]} \\ & & S^4 = \mathbb{HP}^1 \end{array}$$

The total space of this fibration is in fact the standard 7-sphere, but a variation on this construction will give us our result.

Note that since quaternionic multiplication is noncommutative, we have to specify that the equivalence relation for $\mathbb{HP}^1$ uses left multiplication.

As we stated above $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z}^2$ classifies the S^3-bundles over S^4. An explicit isomorphism between these groups is as follows: for $(h, j) \in \mathbb{Z}^2$, let $f_{h,j} : S^3 \to \mathrm{SO}(4)$ be given by $f_{h,j}(u) \cdot v = u^h v u^j$ for $u \in S^3$, $v \in \mathbb{R}^4$, and regarding S^3 as the set of norm 1 quaternions.

To see that the fiber bundles we get from these transition maps are actually distinct (and in fact, all possible S^3-bundles over S^4), we use the classical fact that $S^3 \times S^3$ double covers $\mathrm{SO}(4)$ by the map

$$\begin{aligned} \Psi : S^3 \times S^3 &\to \mathrm{SO}(4) \\ (u, v) &\mapsto \psi_{u,v} = (x \mapsto uxv^{-1}) \end{aligned}$$

Ψ is a group homomorphism with kernel $\{(1, 1), (-1, -1)\}$, so the map is indeed a double cover, so, for one thing, from the homotopy lifting lemma and the mapping lifting theorems we know that $\pi_3(S^3 \times S^3) = \pi_3(\mathrm{SO}(4)) = \mathbb{Z}^2$. Moreover, our $f_{h,j}$ are obtained as the composition of Ψ with $\tilde{f}_{h,j} : S^3 \to S^3 \times S^3$ given by $u \mapsto (u^h, u^{-j})$.

Using 2.3, this extends to a transition map between the North and South hemispheres of our sphere given by

$$(u, v) \mapsto (u', v') = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|} \right)$$

where the map is understood to have domain and codomain $(\mathbb{R}^4 \setminus \{0\}) \times S^3 \ni (u, v)$, and where we freely identify elements of $\mathbb{R}^4$ with elements of $\mathbb{H}$. This gluing map gives us the total space $M_{h,j}$.

Note that $u \mapsto \frac{u}{\|u\|^2}$ is the normal transition map for stereographic projection, i.e, the transition map gluing $\mathbb{R}^4 \setminus \{0\}$ to itself to give S^4.

Since $f_{h,j} = \Psi \circ \tilde{f}_{h,j}$ by construction, we can see that the $f_{h,j}$ are not homotopic since if such a homotopy existed, it would lift to a homotopy from S^3 to $S^3 \times S^3$ by the homotopy lifting property, and the $\tilde{f}_{h,j}$ clearly correspond to different elements of $\pi_3(S^3 \times S^3) \cong \mathbb{Z}^2$.

We call the fiber bundle obtained by this gluing map $\xi_{h,j} : M_{h,j} \to S^4$ with total space $M_{h,j}$.

Topologically Spherical

Taking the standard charts on S^4 as above, when $h + j = 1$, $f : M_{h,j} \to \mathbb{R}$ given by

$$f(u,v) = \frac{\mathrm{Re}(v)}{\sqrt{1+\|u\|^2}} = \frac{\mathrm{Re}(u'(v')^{-1})}{\sqrt{1+\|u'(v')^{-1}\|^2}}$$

will be the Morse function that we desire.

Kuperberg[4] notes that the point of this seemingly unmotivated function is that the real part of a unit length quaternion is preserved under both conjugation (as in aba^{-1}, not as in the quaternionic analogue of complex conjugation) and inversion, so, letting $r = \|u\|$ and $\hat{u} = \frac{u}{r}$, which takes v' to $\hat{u}^h v \hat{u}^j r^{h+j-1} = \hat{u}^h v \hat{u}^j$ by the restriction on h, j. Moreover, $v'\hat{u}^{-1} = \hat{u}^h v \hat{u}^{-h}$ is clearly conjugate to v, and therefore

$$\mathrm{Re}(v) = \mathrm{Re}(v'\hat{u}^{-1}) = \mathrm{Re}(\hat{u}(v')^{-1})$$

Thus, we have

$$f(u,v) = \frac{\mathrm{Re}(v)}{\sqrt{1+r^2}} = \frac{\mathrm{Re}(\hat{u}(v')^{-1})}{\sqrt{1+r^2}} = \frac{\mathrm{Re}(u'(v')^{-1})}{\sqrt{1+r^{-2}}} = \frac{\mathrm{Re}(u'(v')^{-1})}{\sqrt{1+\|u'(v')^{-1}\|^2}}$$

Kuperberg additionally suggests that a possible motivation for this definition is the quaternionic Hopf fibration on the standard S^4, with a Morse function on the total space (the standard S^7) given by one of the coordinates on $\mathbb{R}^8$, which clearly has a unique minimum and maximum on S^7, which (one can imagine) might correspond to taking the real part once we've passed to the unit quaternion picture, and somehow this Morse function survives when $h + j = 1$.

Indeed, in a subsequent paper of Milnor's, the following result (in this vein) is stated and proven:[5]

Proposition 3.1

Let $f : S^m \times S^n \to S^m \times S^n$ be a diffeomorphism, with $f(x,y) = (x', y')$, and M the $m+n+1$ dimensional manifold obtained by gluing $\mathbb{R}^{m+1} \times S^n$ to $S^m \times \mathbb{R}^{n+1}$ along the correspondence $(tx, y) \mapsto (x', t'y')$ where $t' = \frac{1}{t}$.

Suppose $(x,y) \xrightarrow{f} (x',y')$ satisfies $y_n = y'_n$ for all (x,y), where $(y_0, \cdots, y_n)$ and $(y'_0, \cdots, y'_n)$ are the standard coordinates on S^n obtained by the embedding in $\mathbb{R}^{n+1}$. Then the manifold M obtained by gluing along f is homeomorphic to S^{n+1}.

[4] Kuperberg, *Morse Theory and Exotic Spheres*.

We will omit the verification that f as defined satisfies the hypothesis of Reeb's theorem. The justification for f having exactly two critical points is straightforward after noticing that, for fixed u, $\mathrm{Re}(v) = \pm 1$ is required for (u, v) to be a critical point, so we only need to check for criticality at points of the form $(u, \pm 1)$.

[5] Milnor, "Differentiable Structures on Spheres".

The proof uses Reeb's theorem as above, with the function

$$(tx, y) \mapsto \frac{y_n}{\sqrt{1+t^2}} \qquad (x', t'y') \mapsto \frac{t'y'_n}{\sqrt{1+t'^2}}$$

in the first and second coordinate systems respectively, which has exactly two critical points which are non-degenerate. Thus we can see that taking the real part above was somewhat a red herring; we're just using the fact that any of the coordinates on $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a Morse function satisfying the hypothesis of Reeb's theorem.

Trying to parse the meaning of this Morse function was one of the more difficult parts of this paper; Milnor just pulls it out of a hat.

The Pontryagin-Hirzebruch Cabal

The state of affairs is that we have now constructed $M_{h,j}$ for all $h, j \in \mathbb{Z}^2$, and shown that if $h + j = 1$ (and in fact also if $h + j = -1$, as we will see below), $M_{h,j}$ is homeomorphic to S^7. We will now construct some associated 8-manifolds for these $M_{h,j}$ to apply the Hirzebruch signature theorem, and conclude that some of the $M_{h,j}$ cannot be diffeomorphic to the standard S^7.

First, we state the main result that we will need:

Theorem 3.2: Hirzebruch Signature Theorem

Let M be a closed orientable smooth manifold of dimension 8 with signature $\tau(M)$. Then

$$\tau(M) = \frac{1}{45}(7p_2(M) - p_1^2(M))$$

Recall that the signature is the number of positive eigenvalues minus the number of negative eigenvalues of the (symmetric, bilinear) intersection form $H^4(M) \times H^4(M) \to H^8(M)$ given by $(\alpha, \beta) \mapsto \alpha \smile \beta$.

We can't really apply this (quite powerful) theorem yet, since we have only constructed 7-manifolds so far, so to finish the result, we build some associated 8-manifolds and compute some of their characteristic classes.

Lemma 3.3

Let ι be the standard generator for $H^4(S^4)$ (Poincaré dual to the class of a point), then $p_1(\xi_{h,j}) = \pm 2(h-j)\iota \in H^4(S^4)$.

Milnor largely skims over the details in this proof, which I think are fairly nonobvious, to me at least.

Proof : Note that there is an argument that has to be made to even make sense of the Pontryagin classes of sphere bundles (Pontryagin classes being *a priori* only defined for real vector bundles). The argument is this: by the long exact sequence of homotopy groups, $\pi_3(\mathrm{SO}(4)) \cong \pi_4(\mathrm{BSO}(4))$ with $\mathrm{BSO}(4)$ the classifying space of oriented rank 4 vector bundles, so a homotopy class of a map from S^4 to $\mathrm{BSO}(4)$ gives rise to a rank-4 real vector bundle via pullback. Thus we can canonically identify each of our S^3 bundles with a real rank-4 vector bundle, and take Pontryagin classes there instead.

I am also not sure how this type of argument generalizes to (say) the corresponding octonionic construction of exotic 15-spheres; I assume everything works fine if you enforce $\mathrm{SO}(k+1)$ as the structure group as Milnor does here, avoiding the possibility of exotic self-diffeomorphisms of the fibers, especially since Milnor's paper predates Hatcher's proof of the Smale conjecture by over twenty years.

We claim that $p_1(\xi_{h,j})$ is linear in h and j. The idea[6] is that if we have clutching functions f_1, f_2 on S^{n-1} representing fiber bundles E_1, E_2, we can wedge them together to get a clutching function on $S^{n-1} \vee S^{n-1}$ that represents a fiber bundle E_3 on $S^n \vee S^n$ (the suspension of $S^{n-1} \vee S^{n-1}$ is really two copies of S^n glued along an interval, which is homotopy equivalent to $S^n \vee S^n$). Using one's favorite construction of Chern classes (e.g, obstructions), one can show that $c(E_3) = c(E_1) \oplus c(E_2)$.

[6] Kuperberg, *Characteristic classes of sphere bundles over spheres in terms of clutching functions*.

Addition in π_n is modeled by taking the "pinch map" $S^n \to S^n \vee S^n$ that shrinks the equator to a point, and the induced (contravariant) map on H^n takes $a \oplus b$ to $a + b$, and therefore $c(E_3)$ to $c(E_1) + c(E_2)$. Thus, $(h, j) + (h', j') = (h + h', j + j')$ in $\pi_3(\mathrm{SO}(4))$ corresponds to the wedge product of clutching functions $f_{h,j}$, $f_{h',j'}$ and the above argument suffices to show that the Pontryagin class therefore depends linearly on h and j.

The other ingredient we need here is that p_1 is independent of the orientation of the fiber, but if the orientation of S^3 is reversed, $\xi_{h,j}$ goes to $\xi_{-j,-h}$; note that conjugation on S^3 corresponds to negating three of four coordinates (viewed via the standard embedding) which is orientation reversing, and conjugation clearly takes $f_{h,j}$ to $f_{-j,-h}$ (in the explicit equations above) from which the claim follows.

Note that this also implies that $M_{h,j}$ for $h+j = -1$ is topologically a sphere.

Putting these two facts together, and doing some arithmetic, we find that $p_1(\xi_{h,j}) = c(h-j)\iota$ for some constant c. The sketch of the calculation of c is as follows: form the associated 4-cell bundle $\sigma_{h,j} : B_{h,j} \to S^4$ by filling in each fiber with a 4-disk, whose total space is an 8-manifold with boundary $M_{h,j}$. The tangent bundle of $B_{h,j}$ splits naturally as the bundle of vectors tangent to each fiber and those vectors normal to each fiber, and using the formula for the Chern class of a Whitney sum (together with the fact that the tangent bundle of S^4 has $p_1 = 0$), we find that $p_1(B_{h,j}) = \sigma^*_{h,j}(c(h-j)\iota) = c(h-j)\alpha$ where $\alpha = \sigma^*_{h,j}(\iota)$ is the generator of $H^4(B_{h,j})$. There is then some case analysis to show that $c = \pm 2$. ■

What Milnor shows in the original paper is that for any two 8-manifolds with the same (oriented) boundary, $2p_1^2 - \tau \pmod 7$ is conserved. Since the cobordism group of oriented 7-manifolds is trivial (due to Thom), every 7 manifold is a boundary, and this result shows that $2p_1^2 - \tau \pmod 7$ is an invariant of 7-manifold (roughly speaking, since one still has to pick an 8-manifold with specified boundary and then glue it shut somehow, but these choices don't matter). All we do here is simplify this abstraction by constructing a specific closed 8-manifold for our needs.

From here, using the assumption that $M_{h,j}$ is diffeomorphic to the standard 7-sphere (when $h+j = 1$), we may smoothly glue an 8-disk to $B_{h,j}$ to obtain a closed 8-manifold $K_{h,j}$ with $p_1 = \pm 2(h-j)$ (again by a Whitney splitting type argument).

Since $H^4(K_{h,j})$ is just $\mathbb{Z}$ (since gluing an 8-disk won't affect fourth cohomology), the signature is ± 1 since it comes from a 1×1 matrix, so pulling everything together, the Hirzebruch Signature Theorem mod 7 gives us (after some arithmetic)

$$(h-j)^2 \equiv \pm 1 \pmod 7$$

which simplifies to $(h-j)^2 \equiv 1 \pmod 7$ because $\left(\frac{-1}{7}\right) = -1$. Now, for example, $(h, j) = (3, -2)$ satisfies $h + j = 1$ but $25 \not\equiv 1 \pmod 7$, and we have obtained the desired contradiction.

Further Reading

For $n \neq 4$, two homotopy n-spheres are h-cobordant iff they are diffeomorphic by the smooth h-cobordism conjecture, so the smooth structures on spheres form a group under connect sum (isomorphic to the oriented cobordism group of homotopy n-spheres). For S^7, this group is $\mathbb{Z}/28$ (so, in particular, there exists an exotic 7-sphere whose 28-fold connect sum with itself is the standard S^7).

Brieskorn[7] in fact gave an explicit construction of the 28 different smooth structures on S^7 as the subset of $\mathbb{C}^5$ cut out by

$$a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0$$

(which is complex codimension one, and therefore real dimension 8) for $k = 1, \cdots, 28$ intersected with a small S^9 around the origin (making it real dimension 7, as desired). In general, subsets of $\mathbb{C}^n$ cut out by $\sum_{i=1}^n x_i^{k_i} = 0$ after intersecting with a small S^{2n-1} around the origin are called *Brieskorn manifolds* and give other examples of exotic spheres.

[7] Brieskorn, "Beispiele zur Differentialtopologie von Singularitäten."

One can also run this construction with the octonions in place of the quaternions, and this was done by Shimada[8] not long after Milnor's paper was published.

[8] Shimada, "Differentiable structures on the 15-sphere and Pontrjagin classes of certain manifolds".

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